Fermi Liquid Theory

When studying the Hartree-Fock approximation, we introduced the dimensionless parameter

$$r_s \sim \frac{E_c}{E_{kin}} \quad \begin{matrix} \leftarrow \text{interaction energy} \\ \leftarrow \text{kinetic energy} \end{matrix}$$

In real metals, one finds $r_s \sim 2-3$

$\rightarrow$ The electron-electron interaction is not weak.

Experimentally, one often finds at sufficiently low temperatures:

resistivity: $\rho \sim A T^2 + \dots$

heat capacity: $C \sim \frac{\pi^2}{3} k_B \, \rho^*(\mu = E_F) \, T + \underbrace{\beta T^3}_{\text{phonon contribution}}$

magnetic susceptibility: $\chi \sim (g \mu_B)^2 \rho^*(\mu = E_F)$

where $\rho^*(\mu = E_F) = m^* k_F / \pi^2 \hbar^2$ is the density of states at the Fermi level. This is of the same form as in the free electron model except that the mass of the electron has been replaced by an

<u>effective mass m^*</u>

$\rightarrow$ Fermi Liquid theory

The Fermi liquid is an effective low energy theory for an interacting Fermi system in terms of weakly interacting quasiparticles with the same quantum numbers as the free Fermi gas (spin, charge, ...) but effective parameters (effective mass m^*, Fermi Liquid parameters)

- The low-temperature properties of a normal system of interacting electrons with only weak disorder and without spontaneously broken symmetries.
- The low-energy excitations of a Fermi liquid can be understood as those of a diluted gas of fermions, the so-called quasiparticles
- The applicability rests on the assumption that the low-energy excitations are characterized by well-defined energy-momentum relations and that these excitations have small decay rates
- The quasiparticles are characterized by the quantum numbers of a gas of free fermions (no broken symmetry): $\underline{k}$, $\underline{S}$, S^z
- The Fermi liquid is adiabatically connected to the Fermi gas

- non-interacting Green function

consider the Schrödinger equation for a single particle

$$i \frac{\partial \Psi(\underline{r},t)}{\partial t} - H\Psi(\underline{r},t) = 0 \qquad (*)$$

and generalize this equation to

$$i\frac{\partial}{\partial t} G - HG = i\delta(\vec{r}-\vec{r}\,')\delta(t-t')$$

with initial condition $G(\underline{r}, t+0, \underline{r}', t) = \delta(\underline{r}-\underline{r}')$

The Green function represents the probability amplitude for a particle transition from point $\underline{r}'$ at time t' to point $\underline{r}$ at time t $(t>t')$

$$\Psi(\underline{r}, t+\tau) = \int d\underline{r}'\, G(\underline{r}, t+\tau, \underline{r}', t)\, \Psi(\underline{r}', t) \qquad (\square)$$

(causality requires $G=0$ for $\tau<0$).

In terms of the eigenfunctions of the Hamiltonian

$$H\varphi_\lambda(\underline{r}) = \varepsilon_\lambda \varphi_\lambda(\underline{r})$$

we can write

$$G_{\lambda\lambda'}(\tau) = \int d^3r\, d^3r'\, G(r, r'; \tau)\, \varphi_\lambda(\underline{r})\, \varphi^*_{\lambda'}(\underline{r}')$$

In terms of $\varphi_\lambda(r)$, a general solution to (*) can be witten

$$\Psi(\underline{r}', t) = \sum_\lambda c_\lambda \, e^{-i\varepsilon_\lambda t} \varphi_\lambda(\underline{r}')$$

inserting into (□), multiplying by $\varphi_{\lambda'}^*(r)$ and integrating over d^3r yields

$$G_{\lambda\lambda'}(\tau) = G_\lambda(\tau)\,\delta_{\lambda\lambda'} = e^{-i\varepsilon_\lambda \tau}\,\Theta(\tau)$$

Consider now the Fourier transform

$$G_\lambda(\varepsilon) = \frac{1}{i}\int_{-\infty}^{+\infty} d\tau \, e^{i\varepsilon\tau} G_\lambda(\tau)$$

$$= \frac{1}{\varepsilon - \varepsilon_\lambda + i\delta} \qquad \delta \to +0$$

and $$G_\lambda(\tau) = i\int_{-\infty}^{+\infty} \frac{d\varepsilon}{2\pi} \frac{e^{-i\varepsilon\tau}}{\varepsilon - \varepsilon_\lambda + i\delta} = \begin{cases} e^{-i\varepsilon_\lambda \tau} & \text{f. } \tau > 0 \\ 0 & \text{f. } \tau < 0 \end{cases}$$

In a many-fermion system, the definition needs to be slightly generalized due to Pauli's exclusion principle. In addition, various Green functions exist

Basically,
$$G(r,t,r',t') \underset{t>t'}{=} \langle 0 | \hat{\Psi}(rt)\,\hat{\Psi}^+(r',t') | 0 \rangle$$

- non-interacting electrons: $G_0(\underline{k},\omega) = \dfrac{1}{\omega - \varepsilon_k + i\delta}$

- interacting electrons: $G(k,\omega) = \dfrac{1}{\omega - \varepsilon_k - \underbrace{\Sigma(k,\omega)}_{\text{self-energy}}}$

The self-energy has a real and an imaginary component

The Green function of free electrons, $G_0(k,\omega)$, has a pole at $\varepsilon_k = \frac{k^2}{2m} - \mu$. Let us assume that the Green function of the interacting system also has a pole:

$$G(k,\omega) \approx \frac{1}{\omega - \tilde{\varepsilon}_k}$$

where $\tilde{\varepsilon}_k$ is the spectrum of renormalized quasiparticles. For simplicity, neglect $\mathrm{Im}\,\Sigma(\underline{k},\omega)$ for the moment:

$$\omega - \varepsilon_k - \Sigma'(\underline{k},\omega) = 0 \approx \omega - \tilde{\varepsilon}_k \Big|_{\omega = \tilde{\varepsilon}_k}$$

$$\Rightarrow \tilde{\varepsilon}_k - \varepsilon_k - \Sigma'(\underline{k}, \tilde{\varepsilon}_k) = 0$$

and

$$G(k,\omega) = \frac{1}{\omega - \varepsilon_k - \Sigma'(k,\omega)} = \frac{1}{\omega - \varepsilon_k - \Sigma'(k,\tilde{\varepsilon}_k) - \left.\frac{\partial \Sigma'}{\partial \omega}\right|_{\omega=\tilde{\varepsilon}_k}(\omega - \tilde{\varepsilon}_k)}$$

$$= \frac{1}{\omega - \tilde{\varepsilon}_k - \left.\frac{\partial \Sigma'}{\partial \omega}\right|_{\omega=\tilde{\varepsilon}_k}(\omega - \tilde{\varepsilon}_k)} = \frac{1}{\omega - \tilde{\varepsilon}_k}\left(1 - \left.\frac{\partial \Sigma'}{\partial \omega}\right|_{\omega=\tilde{\varepsilon}_k}\right)^{-1}$$

$$= \frac{Z_k}{\omega - \tilde{\varepsilon}_k}$$

- $Z_k = \left(1 - \left.\frac{\partial \Sigma'}{\partial \omega}\right|_{\omega=\tilde{\varepsilon}_k}\right)^{-1}$ is the residue of the quasiparticle pole; Z_k is also called wave function renormalization factor

- $\lim_{\delta \to 0} \frac{1}{\pi} \operatorname{Im} G(k, \omega + i\delta)$ is called spectral density

It is a quantity that can be measured using STM or ARPES

The resulting spectral density is $A(k,\omega) = Z_k \delta(\omega - \tilde{\varepsilon}_k)$ as we neglected $\operatorname{Im} \Sigma(\underline{k},\omega)$

On general grounds, $\int_{-\infty}^{+\infty} d\varepsilon\, A(\underline{k},\varepsilon) = 1$ for any fermionic spectral density.

In general $Z_k \leq 1$ and $Z_k = 1$ only for the free Fermi gas

In a system with interactions, the quasiparticle contribution to $A(k,\omega)$ is slightly suppressed due to the appearance of an additional "multi-particle" incoherent contribution to $A(\underline{k},\omega)$.

- finite damping will result in a finite width of the quasiparticle peak
- Suppose that the spectrum of "renormalized" quasiparticles can be described by an effective mass approximation:

$$\tilde{\varepsilon}_k = \frac{k^2}{2m^*} - \mu = \varepsilon_k + \Sigma'(k, \tilde{\varepsilon}_k)$$

$$\Rightarrow \frac{1}{2m^*} = \frac{\partial \tilde{\varepsilon}_k}{\partial (k^2)} = \frac{\partial \varepsilon_k}{\partial (k^2)} + \frac{\partial \Sigma'(\underline{k}, \tilde{\varepsilon}_k)}{\partial (k^2)} + \frac{\partial \Sigma'(\underline{k}, \tilde{\varepsilon}_k)}{\partial \tilde{\varepsilon}_k} \frac{\partial \tilde{\varepsilon}_k}{\partial (k^2)}$$

$$= \frac{1}{2m} + \frac{\partial \Sigma'}{2m\, \partial\left(\frac{k^2}{2m}\right)} + \left.\frac{\partial \Sigma'}{\partial \varepsilon}\right|_{\varepsilon = \tilde{\varepsilon}_k} \underbrace{\frac{\partial \tilde{\varepsilon}_k}{\partial (k^2)}}_{\frac{1}{2m^*}}$$

$$\Rightarrow \frac{1}{m^*}\left(1 - \left.\frac{\partial \Sigma'}{\partial \varepsilon}\right|_{\varepsilon = \tilde{\varepsilon}_k}\right) = \frac{1}{m}\left(1 + \frac{\partial \Sigma'}{\partial \varepsilon_k}\right)$$

97

so that

$$\frac{m^*}{m} = \frac{1 - \left.\frac{\partial \Sigma}{\partial \varepsilon}\right|_{\varepsilon = \tilde{\varepsilon}_k}}{1 + \frac{\partial \Sigma}{\partial \varepsilon_k}} = \frac{1}{Z_k} \frac{1}{1 + \frac{\partial \Sigma}{\partial \varepsilon_k}}$$

which gives us an important relation between "mass renormalization" $\frac{m^*}{m}$ and the residue at the quasiparticle pole, i.e. Z_k

When the selfenergy has no dependence on $\underline{k}$ (or equivalently on $\tilde{\varepsilon}_k$), one finds $\frac{m^*}{m} = Z_k^{-1}$ but it is important to remember that m^* is a dynamical quantity.

So far, we have neglected any damping due to $\mathrm{Im}\, \Sigma(k,\omega)$. In general, we expect that the interacting Green function with damping

$$G(k,\omega) = \frac{Z_k}{\omega - \xi(k) + i\underbrace{\gamma(k)}_{\text{damping}}}$$

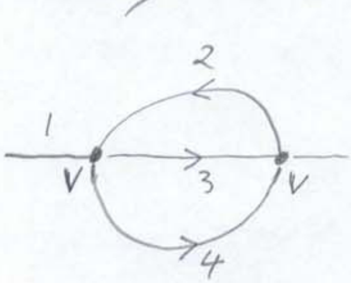
such that the Fourier transform ($\tau > 0$)

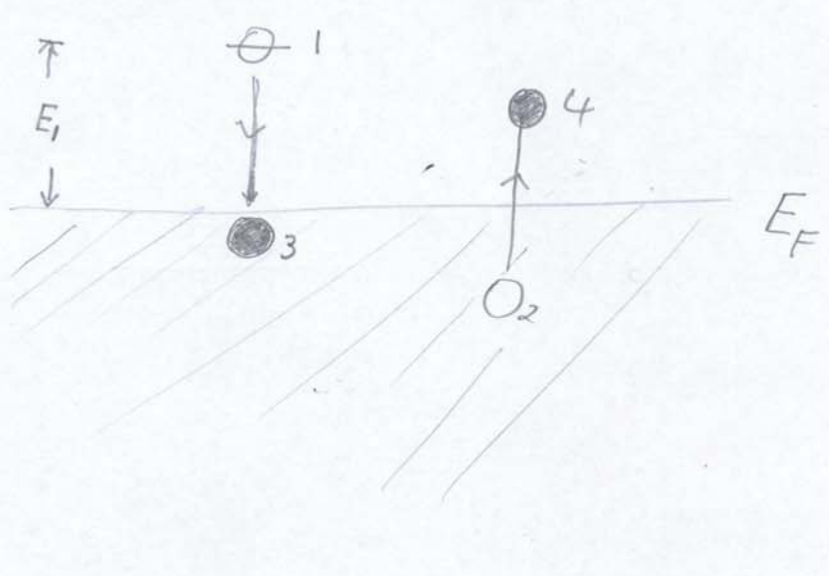
$$G(k,\tau) \approx Z\, e^{-i\xi(k)\tau - \gamma(k)\tau} + \ldots$$

where $\xi(\underline{k}) = \varepsilon(\underline{k}) - E_F$

we have to require that $\gamma(\underline{k}) \ll \xi(\underline{k})$, i.e., weak damping, to make the quasiparticles well-defined.
The existence of damping is due to the fact that the quasiparticle states are not time-independent, i.e., are not exact eigenstates of the underlying Hamiltonian.

Consider a quasiparticle at energy E_1 above the Fermi energy. Due to the interaction with quasiparticles in the Fermi sea, particle-hole excitations are created and the quasiparticle transitions into a new state, that is, the quasiparticle decays.
The decay rate of such a process can be estimated:





99

The decay rate is

$$\frac{1}{\tau_1} = \int dE_2\, N(E_2) \int dE_3\, N(E_3) \int dE_4 N(E_4)\, |V|^2$$

$$\underbrace{f(E_2)}_{\text{occupied}}\ \underbrace{f(-E_3)\, f(-E_4)}_{\text{empty}}\ \underbrace{\delta(E_1 + E_2 - E_3 - E_4)}_{\text{energy conservation}}$$

$$E_2 = E_3 + E_4 - E_1 \rightsquigarrow f(E_3 + E_4 - E_1)$$
$$\rightsquigarrow E_4 < E_1 - E_3$$

$$\approx N_F^{\,3} |V|^2 \int_0^{E_1} dE_3 \int_0^{E_1 - E_3} dE_4 = \tfrac{1}{2} N_F^{\,3} |V|^2 E_1^{\,2}$$

and using that $N_F |V| \gtrsim 1$ and $N_F \sim \frac{1}{E_F}$ one finds

$$\boxed{\frac{1}{\tau_1} \ll E_1}$$

In other words, the broadening of the quasiparticle energies due to the finite lifetime is negligible for small energies E and vanishes for $E \to 0$ (that is $E \to E_F$).

The basic assumption of the Landau Fermi liquid is the existence of a well-defined Fermi surface with the Fermi momentum p_F defined by the usual relation

$$n = \frac{N}{V} = \frac{p_F^{\,3}}{3\pi^2 \hbar^3}$$

where n is the full (interacting) particle density.

It is important to realize that the Fermi liquid is not the only possible ground state of a system of interacting fermions. For example, in the superconducting state there is no Fermi surface in the usual sense due to the energy gap and Luttinger's theorem (above) does not apply.

The basic physical reason for the existence of the Landau Fermi liquid is the Pauli principle that leads to strong restrictions of the phase space close to the vicinity of the Fermi surface.

For finite temperatures the "smearing" of the Fermi distribution $\sim T$ leads to damping due to thermally excited quasiparticles $\sim \frac{T^2}{E_F}$.

We therefore can estimate

$$\boxed{\frac{1}{\tau} \approx \max\left(\frac{\varepsilon^2}{E_F}, \frac{T^2}{E_F}\right)}.$$

Using the Drude expression for the conductivity

$$\sigma = \frac{ne^2}{m}\tau$$

and taking $\tau^{-1} = A\frac{T^2}{E_F}$, we obtain

$$R = \frac{1}{\sigma} \sim \frac{T^2 m}{E_F \, p_F^{\,3} e^2} = \frac{1}{e^2 p_F}\left(\frac{T}{E_F}\right)^2$$

This gives a characteristic temperature dependence $R \sim T^2$ of the resistivity due to electron-electron scattering.

Typically, $p_F \sim \frac{\hbar}{a}$ $\longrightarrow$ $R \sim \frac{\hbar}{e^2 a}\left(\frac{T}{E_F}\right)^2$ gives a tiny contribution to R which is usually masked by other scattering mechanisms (like phonons) except for lowest temperatures.

$\uparrow$ interatomic spacing

We have shown, that the quasi-particle damping

$$\frac{1}{\tau} \sim \frac{\varepsilon^2}{E_F}$$

in lowest order.

Higher order processes with a larger number of excited particle-hole pairs turn out to be proportional to higher powers of ε.

In a homogeneous, isotropic Fermi liquid, the value of $\mathrm{Re}\,\Sigma(\underline{k},\omega) = \Sigma'(\underline{k},\omega)$ depends only on the modulus of momentum $k = |\underline{k}|$

The Fermi momentum k_F in the interacting system obeys

$$\frac{k_F^2}{2m} + \Sigma(k_F, 0) = \mu$$

Expanding $\Sigma(k,\omega)$ in terms of $k-k_F$ yields an expression for $G(k,\omega)$ that is valid close to the Fermi surface.

$$\left.\frac{\partial\Sigma}{\partial k}\right|_F \equiv \left.\frac{\partial\Sigma(k,\omega)}{\partial k}\right|_{k=k_F,\,\omega\to 0}$$

$$G^{-1}(k,\omega) \approx \omega - \frac{k^2}{2m} + \mu - \Sigma(k_F,0) - \left.\frac{\partial\Sigma}{\partial k}\right|_F (k-k_F)$$
$$- \left.\frac{\partial\Sigma}{\partial\omega}\right|_F \omega + i\tilde{\alpha}|k-k_F|(k-k_F)$$

$$= \left[1 - \left.\frac{\partial\Sigma}{\partial\omega}\right|_F\right]\omega - \left[\frac{k_F}{m} + \left.\frac{\partial\Sigma}{\partial k}\right|_F\right](k-k_F) + i\tilde{\alpha}|k-k_F|(k-k_F)$$

where $\frac{1}{\tau} \sim \mathrm{Im}\,\Sigma \sim (k-k_F)^2$ was used and the sign change of the imaginary part of the Green function at $\omega = 0$ was incorporated.

→ The Green function for an interacting system of fermions can be written close to the Fermi surface as

$$G(k,\omega) = \frac{Z}{\omega - v_F(k-k_F) + i\alpha|k-k_F|(k-k_F)} + G^{regular}(k,\omega) \qquad (\Delta)$$

where we used

$$Z^{-1} = 1 - \left.\frac{\partial\Sigma}{\partial\omega}\right|_F = \left.\frac{\partial G^{-1}}{\partial\omega}\right|_F$$ for the residue of the quasiparticle pole

and

$$v_F = \frac{\frac{k_F}{m} + \left.\frac{\partial\Sigma}{\partial k}\right|_F}{\left.\frac{\partial G^{-1}}{\partial\omega}\right|_F} = -\frac{\left.\frac{\partial G^{-1}}{\partial k}\right|_F}{\left.\frac{\partial G^{-1}}{\partial\omega}\right|_F}$$ for the velocity at the Fermi surface

and $\alpha = Z\tilde{\alpha}$ and $G^{regular}(k,\omega)$ is some regular (non-singular) part with no poles close to the Fermi surface.

(Δ) leads to a <u>discontinuity</u> in particle distribution in momentum space (at $T=0$):

⇝ calculate $n(k)$ on both sides of the Fermi surface

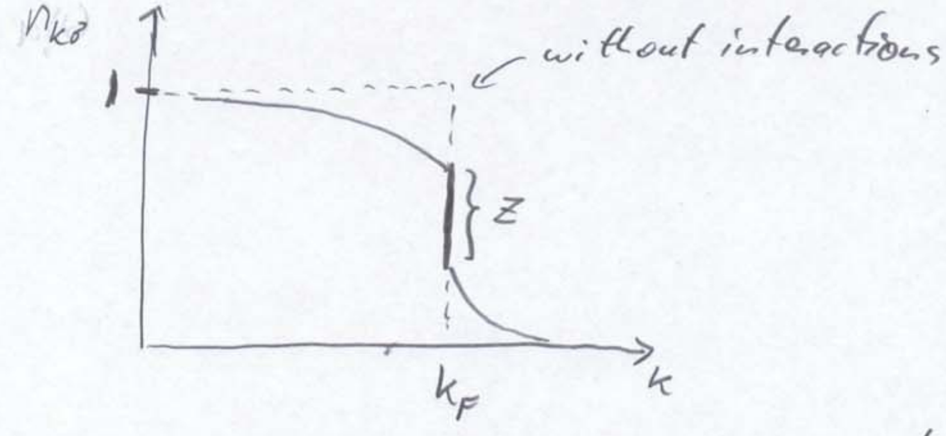
$$\lim_{q\to 0}\left(n(k_F+q) - n(k_F-q)\right) \quad \text{where} \quad n(k) = -i\lim_{t\to -0}\int_{-\infty}^{+\infty}\frac{d\varepsilon}{2\pi}e^{-i\varepsilon t}G(k,\varepsilon)$$

as $G^{regular}(k,\omega)$ is regular, its contribution will vanish as $q\to 0$

$$n(k_F-q)-n(k_F+q) = -i\int_{-\infty}^{+\infty}\frac{d\varepsilon}{2\pi}\left\{\frac{Z}{\varepsilon+v_F q-i\delta}-\frac{Z}{\varepsilon-v_F q+i\delta}\right\}$$

where $\frac{1}{\tau}\sim(k-k_F)^2 \to \mathrm{sgn}(k-k_F)=\mathrm{sgn}(\varepsilon)$ was used for a quasiparticle pole at the Fermi surface

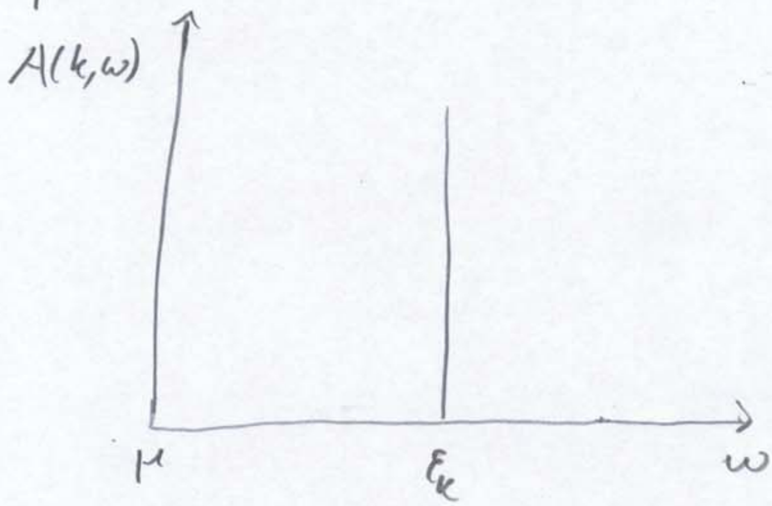
Therefore, $n(k_F-0)-n(k_F+0)=Z$



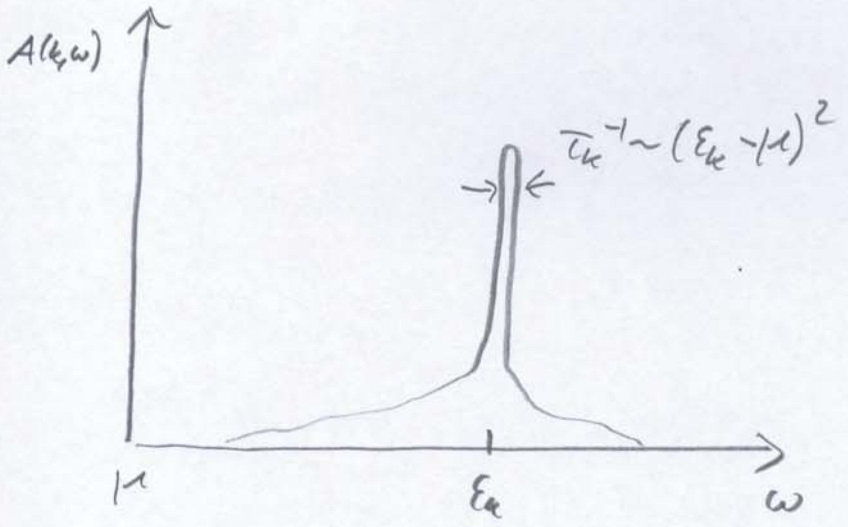
Qualitative form of the particle distribution function in the Fermi liquid at $T=0$

The existence of a discontinuity in $n_{k\sigma}$ allows a strict definition of the Fermi surface in a system of interacting fermions.

spectral density of the Fermi gas



spectral density of the Fermi liquid



- The spectral density of electrons can be measured via angle resolved photoemission spectroscopy (ARPES)
- $Z \to 0$: non-Fermi liquid state
  as e.g. in one spatial dimension
  (Luttinger liquid)