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Quantum Many-Body Systems

2. Exercise Sheet

Mini-Review: The time-evolution operator

Before we ask you to perform any calculations we will briefly review the properties of the time evolution operator in quantum mechanics. In Dirac ket notation, one writes the Schrödinger equation as

$$i\hbar\frac{\partial}{\partial t}|\psi_t\rangle = \hat{H}(t)|\psi_t\rangle, \tag{4.1}$$

where the Hamiltonian $\hat{H}(t)$ may have an explicit time dependence. Introducing the time-evolution operator $\hat{U}(t,t')$ (with $t > t'$), through the relation $|\psi_t\rangle = \hat{U}(t,t')|\psi_{t'}\rangle$, one sees by direct substitution in (4.1) that $\hat{U}(t,t')$ obeys the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t,t') = \hat{H}(t)\hat{U}(t,t'), \tag{4.2}$$

with the boundary condition $\hat{U}(t',t') = 1$, since $\lim_{t\to t'}|\psi_t\rangle = |\psi_{t'}\rangle$ by continuity of the wave-function in time. Also, for any $t > u > t'$, one has that $|\psi_t\rangle = \hat{U}(t,u)|\psi_u\rangle$ and $|\psi_u\rangle = \hat{U}(u,t')|\psi_{t'}\rangle$ and thus substituting the latter equation in the former, we conclude that

$$\hat{U}(t,t') = \hat{U}(t,u)\hat{U}(u,t'), \tag{4.3}$$

with $t > u > t'$. This is known as the *semi-group* property of $\hat{U}(t,t')$. The equation (4.2) is equivalent to the integral equation

$$\hat{U}(t,t') = 1 - \frac{i}{\hbar}\int_{t'}^{t} du\, \hat{H}(u)\hat{U}(u,t'), \tag{4.4}$$

which already takes into account the boundary condition at $t = t'$. Substituting the expression for $\hat{U}(u,t')$ as given by (4.4) in (4.4) itself and iterating the resulting equation, one sees that $\hat{U}(t,t') = \sum_{n=0}^{\infty} \hat{U}_n(t,t')$ where $\hat{U}_n(t,t')$ is given by

$$\begin{aligned}\hat{U}_n(t,t') &= \frac{(-i)^n}{\hbar^n}\int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \ldots \int_{t'}^{t_{n-1}} dt_n\, \hat{H}(t_1)\hat{H}(t_2)\ldots\hat{H}(t_n) &(4.5)\\ &= \frac{(-i)^n}{\hbar^n}\int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 \ldots \int_{t'}^{t} dt_n\, \hat{H}(t_1)\hat{H}(t_2)\ldots\hat{H}(t_n)\Theta(t_1-t_2)\ldots\Theta(t_{n-1}-t_n),\end{aligned}$$

where we have used the Θ-functions to rewrite all integrals with the same upper limit. Once can now permute the n dummy indices $t_1,\ldots,t_n$, provided one divides the end result by the total number of permutations, $n!$. One obtains

$$\begin{aligned}\hat{U}(t,t') &= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!\hbar^n}\int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \ldots \int_{t'}^{t_{n-1}} dt_n\; T\{\hat{H}(t_1)\hat{H}(t_2)\ldots\hat{H}(t_n)\} &(4.6)\\ &= T\exp\Big(-\frac{i}{\hbar}\int_{t'}^{t} du\,\hat{H}(u)\Big), &(4.7)\end{aligned}$$

where T is the time-ordering operator that orders the Hamiltonian operator at different times in chronological order. The time-ordered exponential in (4.7) is defined in terms of the series in (4.6) and is merely a compact way of writing this series. If $\hat{H}(u) = \hat{H}_0$, which is time-independent, then the time-ordering in (4.6) becomes a simple product and one may sum the series, obtaining

$$\hat{U}(t,t') = \exp\Big(-\frac{i}{\hbar}\hat{H}_0(t-t')\Big), \tag{4.8}$$

which is only dependent on the difference $t - t'$. Note, as an aside, that if one can write the Hamiltonian as $\hat{H}(t) = \hat{H}_0 + \hat{V}(t)$, where the quantum dynamical problem defined by $\hat{H}_0$ is solvable and $\hat{V}(t)$ is an interaction term, eventually dependent on time in the Schrödinger representation, one may define the time-evolution in the interaction representation by $\hat{S}(t,t') = e^{i\hat{H}_0 t/\hbar}\hat{U}(t,t')e^{-i\hat{H}_0 t/\hbar}$. The equation obeyed by $\hat{S}(t,t')$ is given, using (4.2), by

$$i\hbar\frac{\partial}{\partial t}\hat{S}(t,t') = \hat{V}^I(t)\hat{S}(t,t'), \tag{4.9}$$

where $\hat{V}^I(t) = e^{i\hat{H}_0 t/\hbar}\hat{V}(t)e^{-i\hat{H}_0 t/\hbar}$ is the interaction term in the interaction representation. Note that even if $\hat{V}(t) = \hat{V}$ is time-independent in the Schrödinger representation, it will in general be explicitly time-dependent in the interaction representation (unless

$[\hat{H}_0, \hat{V}]_- = 0$, but in that case these two operators have common eigenstates and the problem is solved as soon as we know the eigenstates of $\hat{H}_0$). One can now repeat the analysis above, obtaining

$$\hat{S}(t,t') = T\exp\Big(-\frac{i}{\hbar}\int_{t'}^{t} du\, \hat{V}^I(u)\Big), \tag{4.10}$$

with the same boundary condition and with the *semi-group* property also holding for $\hat{S}(t,t')$. This is the expression obtained in the lecture and ends our crash course.

5 Green function for the one-particle Schrödinger equation

(a) For a single-particle problem, one defines the wave-function of such a particle in real space by $\psi(\boldsymbol{r},t) = \langle \boldsymbol{r}|\psi_t\rangle$. Given that $\psi(\boldsymbol{r},t) = \langle \boldsymbol{r}|\hat{U}(t,0)|\psi_0\rangle$ where $|\psi_0\rangle$ is the wave-function at $t=0$, show that $\psi(\boldsymbol{r},t)$ satisfies the integral equation

$$\psi(\boldsymbol{r},t) = i\hbar \int d^3r' G^R(\boldsymbol{r},t;\boldsymbol{r}',t')\psi(\boldsymbol{r}',t), \tag{5.1}$$

with $t > t'$ and where $G^R(\boldsymbol{r},t;\boldsymbol{r}',t') = -\frac{i}{\hbar}\langle \boldsymbol{r}|\hat{U}(t,t')|\boldsymbol{r}'\rangle\Theta(t-t')$ is the Green function (also known as the propagator) of the Schrödinger equation.
(2 points)

Hint: Use the semi-group property (4.3) and the completeness of the basis of the position eigenstates, *i.e.* the relation $\int d^3r'|\boldsymbol{r}'\rangle\langle\boldsymbol{r}'| = \mathbf{1}$.

(b) Consider now the case in which the Hamiltonian does not depend on time and thus $\hat{U}(t,t')$ reduces to (4.8). Show that in such a case, one may write

$$G^R(\boldsymbol{r},t-t';\boldsymbol{r}',0) = -\frac{i}{\hbar}\sum_\alpha \phi_\alpha(\boldsymbol{r})\phi_\alpha^*(\boldsymbol{r}')e^{-i\varepsilon_\alpha(t-t')/\hbar}\Theta(t-t'), \tag{5.2}$$

where $\phi_\alpha(\boldsymbol{r}) = \langle \boldsymbol{r}|\phi_\alpha\rangle$ and $\phi_\alpha\rangle$ are the eigenstates of $\hat{H}_0$ with ε_α being the corresponding eigen-energies, *i.e.* $\hat{H}_0|\phi_\alpha\rangle = \varepsilon_\alpha|\phi_\alpha\rangle$. This is known as the *spectral representation* of the Green function.
(2 points)
Hint: Use the completeness of the eigenstates of the Hamiltonian, *i.e.*, the relation $\sum_\alpha |\phi_\alpha\rangle\langle\phi_\alpha| = \mathbf{1}$.

(c) Show that the Fourier transform of (5.2) is given by

$$G^R(\boldsymbol{r},\boldsymbol{r}',\omega) = \frac{i}{\hbar}\sum_\alpha \frac{\phi_\alpha(\boldsymbol{r})\phi_\alpha(\boldsymbol{r}')}{\omega-\omega_\alpha+i\varepsilon}, \tag{5.3}$$

where $\omega_\alpha = \varepsilon_\alpha/\hbar$ and where the infinitesimal factor $i\varepsilon$ was introduced in the exponent of the Fourier transform in order to insure convergence.

(d) Show that $\rho(\omega) = -\frac{\hbar}{\pi}\int d^3r\, \mathrm{Im} G^R(\boldsymbol{r}, \boldsymbol{r}, \omega)$ is given by

$$\rho(\omega) = \sum_\alpha \delta(\omega - \omega_\alpha). \tag{5.4}$$

This quantity is known as the *density of states* of the Hamiltonian, since the integration of $\rho(\omega)$ within a given frequency interval counts the number of states inside that interval. If the spectrum is discrete, such density of states will be a sum of delta-peaks, but it may have an analytic form in the case of a continuous spectrum. **(2 points)**
Hint: Use the definition of $\rho(\omega)$, the fact that $|\phi_\alpha\rangle$ are normalized states and the Dirac relation $\dfrac{1}{x+i\varepsilon} = P\big(\dfrac{1}{x}\big) - i\pi\delta(x)$.

6 Time-evolution of annihilation and creation operators in a many-body system described by a quadratic Hamiltonian and Wick's theorem at finite temperatures

(a) Derive the equation of motion for the annihilation and creation operators in the Heisenberg representation in imaginary time for a system of independent bosons (fermions) whose Hamiltonian is given by

$$\hat{H}_0 = \sum_\nu \varepsilon_\nu \hat{c}_\nu^\dagger \hat{c}_\nu, \tag{6.1}$$

where the operators obey commutation (anti-commutation) relations given by (0.10) and (0.15) (see exercise sheet 1). Note that these operators are given in the Heisenberg representation in imaginary time by

$$\hat{c}_\alpha(\tau) = e^{\hat{H}_0\tau}\hat{c}_\alpha e^{-\hat{H}_0\tau}, \tag{6.2}$$

$$\hat{c}_\alpha^\dagger(\tau) = e^{\hat{H}_0\tau}\hat{c}_\alpha^\dagger e^{-\hat{H}_0\tau}. \tag{6.3}$$

Show that these equations have the solution

$$\hat{c}_\alpha(\tau) = \hat{c}_\alpha e^{-\varepsilon_\alpha\tau}, \tag{6.4}$$

$$\hat{c}_\alpha^\dagger(\tau) = \hat{c}_\alpha^\dagger e^{\varepsilon_\alpha\tau}. \tag{6.5}$$

Do note that in imaginary time $\hat{c}_\alpha(\tau)$ and $\hat{c}_\alpha^\dagger(\tau)$ are not each other's adjoint, except at $\tau = 0$. However, if one makes the analytic continuation $\tau \to it/\hbar$, one does obtain the correct time evolution of these operators in real time.
(3 points)
Hint: Solve explicitly the differential equation for these operators as an initial value problem (one knows to which operators $\hat{c}_\alpha(\tau)$ and $\hat{c}_\alpha^\dagger(\tau)$ have to reduce at $\tau = 0$).

(b) We defined $\langle \hat{A} \rangle = \dfrac{1}{Z_0} \mathrm{Tr}\big(\hat{A} e^{-\beta(\hat{H}_0 - \mu \hat{N})}\big)$. Show that if $\hat{H}_0$ is given by (6.1), $\langle \hat{c}^\dagger_\alpha \hat{c}_\gamma \rangle = n_\alpha \delta_{\alpha,\gamma}$, where $n_\alpha = \dfrac{1}{e^{\beta(\varepsilon_\alpha - \mu)} - \xi}$, with $\xi = \pm 1$, for bosons or fermions. These functions correspond, respectively, to the Bose-Einstein and Fermi-Dirac distributions for bosons or fermions. Show that $\langle \hat{c}_\gamma \hat{c}^\dagger_\alpha \rangle = (1 + \xi n_\gamma)\delta_{\alpha,\gamma}$.
(2 points)
Hint: Use the (anti-)commutation relations between the annihilation and creation operators to interchange them inside the average and use (6.2) to (6.5) to show the identities $\hat{c}_\gamma \, e^{-\beta(\hat{H}_0 - \mu \hat{N})} = e^{-\beta(\hat{H}_0 - \mu \hat{N})} \hat{c}_\gamma e^{-\beta(\varepsilon_\gamma - \mu)}$ and
$\hat{c}^\dagger_\gamma \, e^{-\beta(\hat{H}_0 - \mu \hat{N})} = e^{-\beta(\hat{H}_0 - \mu \hat{N})} \hat{c}^\dagger_\gamma e^{\beta(\varepsilon_\gamma - \mu)}$. Note that the operator $\hat{H}_{\mathrm{GC}} = \hat{H}_0 - \mu \hat{N}$, where $\hat{N} = \sum_\nu \hat{c}^\dagger_\nu \hat{c}_\nu$, is of the same form as $\hat{H}_0$, it simply involves renormalised energies $\varepsilon_\nu = \varepsilon_\nu - \mu$. Finally, use the cyclic invariance of the trace.

(c) Following the same steps as in the previous exercise, show that

$$\langle \hat{c}^\dagger_\alpha \hat{c}_\gamma \hat{c}^\dagger_\chi \hat{c}_\eta \rangle = n_\alpha n_\chi \delta_{\alpha,\gamma} \delta_{\chi,\eta} + n_\alpha \big(1 + \xi n_\gamma\big) \delta_{\alpha,\eta} \delta_{\gamma,\chi}. \tag{6.6}$$

(2 points)

(d) Finally, consider the general case of $\langle \hat{A}_1 \hat{A}_2 \dots \hat{A}_N \rangle$ where each $\hat{A}_i$ $(i = 1, \dots, N)$ is either an annihilation or creation operator in the basis that diagonalizes $\hat{H}_0$, which is of the form (6.1). Thus, the (anti-)commutator $[\hat{A}_i, \hat{A}_j]_{-\xi}$ for arbitrary i and j is a c-number. Show that

$$\langle \hat{A}_1 \hat{A}_2 \dots \hat{A}_N \rangle = \frac{1}{1 - \xi^{N-1} e^{\mp \beta(\varepsilon_1 - \mu)}} \sum_{i=2}^{N} \xi^{i-2} [\hat{A}_1, \hat{A}_i]_{-\xi} \, \langle \hat{A}_2 \dots \hat{A}_{i-1} \hat{A}_{i+1} \dots \hat{A}_N \rangle, \tag{6.7}$$

where the $\mp$ sign depends on whether $\hat{A}_1$ is, respectively, an annihilation or creation operator. Note that the averages appearing on the right-hand side involve two annihilation or creation operators less and are of the same form as the one we started from. We can thus iterate this formula until we have reduced it to products of commutators and functions of the type $\dfrac{1}{1 - \xi^{M-1} e^{\mp \beta(\varepsilon_i - \mu)}}$. This constitutes Gaudin's proof of Wick's theorem at finite temperature and can even be generalized to the case where the operators $\hat{A}_i$ are given in the interaction representation.
(4 points)

7 Retarded many-body Green function for a boson system with a quadratic Hamiltonian

Take the Hamiltonian describing the many-particle system to be of the form (6.1). Show that the retarded many-body Green function $G^R(\boldsymbol{r}, t; \boldsymbol{r}', t') = -\frac{i}{\hbar} \langle [\hat{\psi}(\boldsymbol{r}, t), \hat{\psi}^\dagger(\boldsymbol{r}', t')]_- \rangle \Theta(t - t')$, where $\hat{\psi}(\boldsymbol{r}, t) = e^{i\hat{H}_0 t/\hbar} \hat{\psi}(\boldsymbol{r}) e^{-i\hat{H}_0 t/\hbar}$, $\hat{\psi}^\dagger(\boldsymbol{r}, t) = e^{-i\hat{H}_0 t/\hbar} \hat{\psi}^\dagger(\boldsymbol{r}) e^{i\hat{H}_0 t/\hbar}$, is exactly given

by the expression (5.2).
(4 points)
Hint: Express the operators $\hat{\psi}(\boldsymbol{r},t)$ and $\hat{\psi}^\dagger(\boldsymbol{r},t)$ in terms of the operators that diagonalize the Hamiltonian $\hat{H}_0$, $\hat{\psi}(\boldsymbol{r},t) = \sum_\alpha \phi_\alpha(\boldsymbol{r})\hat{c}_\alpha(t)$ and $\hat{\psi}^\dagger(\boldsymbol{r},t) = \sum_\alpha \phi^*_\alpha(\boldsymbol{r})\hat{c}^\dagger_\alpha(t)$ and use their known time-evolution to compute their commutator.

8 Free-fermion observables expressed in terms of particle-hole operators and ground-state properties of the Fermi gas

The ground state of a free-fermion system with N fermions of spin 1/2, described by the Hamiltonian $\hat{H}_0 = \sum_{\boldsymbol{k},\sigma} \varepsilon_{\boldsymbol{k}} \hat{c}^\dagger_{\boldsymbol{k},\sigma}\hat{c}_{\boldsymbol{k},\sigma}$ with $\varepsilon_{-\boldsymbol{k}} = \varepsilon_{\boldsymbol{k}}$ is given by $|\phi_0\rangle = \prod_{|\boldsymbol{k}|<k_F,\sigma} \hat{c}^\dagger_{\boldsymbol{k},\sigma}|0\rangle$, which is a Slater determinant, with all states with momentum less than k_F occupied with two fermions (of opposite spin) and all states with momentum greater than k_F being left empty. The value of k_F is chosen so that the number of particles in the system is equal to N. This state is not the vacuum for all of the annihilation operators $\hat{c}_{\boldsymbol{k},\sigma}$, *e.g.* if $|\boldsymbol{k}| < k_F$ then $\hat{c}_{\boldsymbol{k},\sigma}|\phi_0\rangle \neq 0$. It is thus convenient to define a set of annihilation or creation operators such that $|\phi_0\rangle$ acts as the vacuum of such a set of operators. The particle-hole creation and annihilation operators are defined through

$$\hat{\alpha}_{\boldsymbol{k},\sigma} = \hat{c}_{\boldsymbol{k},\sigma}, \quad \hat{\alpha}^\dagger_{\boldsymbol{k},\sigma} = \hat{c}^\dagger_{\boldsymbol{k},\sigma} \quad \text{if } |\boldsymbol{k}| > k_F, \tag{8.1}$$

$$\hat{\beta}_{\boldsymbol{k},\sigma} = \hat{c}^\dagger_{-\boldsymbol{k},-\sigma}, \quad \hat{\beta}^\dagger_{\boldsymbol{k},\sigma} = \hat{c}_{-\boldsymbol{k},-\sigma} \quad \text{if } |\boldsymbol{k}| > k_F. \tag{8.2}$$

(a) Show that the new operators obey the correct anticommutation relations for fermions, *i.e.*, that the transformation defined above is canonical.
(2 points)

(b) Show that $\hat{\alpha}_{\boldsymbol{k},\sigma}|\phi_0\rangle = 0$ and $\hat{\beta}_{\boldsymbol{k},\sigma}|\phi_0\rangle = 0$, thus proving that the $|\phi_0\rangle$ is the vacuum for this set of operators.
(2 points)

(c) Show that one can express the operators of the total energy, momentum, and projection of the spin along the z-axis for a free-fermion system as

$$\hat{N} = N + \sum_{|\boldsymbol{k}|>k_F,\sigma} \hat{\alpha}^\dagger_{\boldsymbol{k},\sigma}\hat{\alpha}_{\boldsymbol{k},\sigma} - \sum_{|\boldsymbol{k}|\le k_F,\sigma} \hat{\beta}^\dagger_{\boldsymbol{k},\sigma}\hat{\beta}_{\boldsymbol{k},\sigma} \tag{8.3}$$

$$\hat{H}_0 = E_0 + \sum_{|\boldsymbol{k}|>k_F,\sigma} \varepsilon_{\boldsymbol{k}}\, \hat{\alpha}^\dagger_{\boldsymbol{k},\sigma}\hat{\alpha}_{\boldsymbol{k},\sigma} - \sum_{|\boldsymbol{k}|\le k_F,\sigma} \varepsilon_{\boldsymbol{k}}\, \hat{\beta}^\dagger_{\boldsymbol{k},\sigma}\hat{\beta}_{\boldsymbol{k},\sigma} \tag{8.4}$$

$$\hat{\boldsymbol{P}} = \sum_{|\boldsymbol{k}|>k_F,\sigma} \hbar\boldsymbol{k}\, \hat{\alpha}^\dagger_{\boldsymbol{k},\sigma}\hat{\alpha}_{\boldsymbol{k},\sigma} + \sum_{|\boldsymbol{k}|\le k_F,\sigma} \hbar\boldsymbol{k}\, \hat{\beta}^\dagger_{\boldsymbol{k},\sigma}\hat{\beta}_{\boldsymbol{k},\sigma} \tag{8.5}$$

$$S^z = \frac{\hbar}{2}\Big[\sum_{|\boldsymbol{k}|>k_F,\sigma} \sigma\, \hat{\alpha}^\dagger_{\boldsymbol{k},\sigma}\hat{\alpha}_{\boldsymbol{k},\sigma} + \sum_{|\boldsymbol{k}|\le k_F,\sigma} \sigma\, \hat{\beta}^\dagger_{\boldsymbol{k},\sigma}\hat{\beta}_{\boldsymbol{k},\sigma}\Big], \tag{8.6}$$

where $N = \sum_{\boldsymbol{k}\leq k_F,\sigma} 1$ is the total number of particles in the ground state (this expression actually fixes the value of k_F) and $E_0 = \sum_{\boldsymbol{k}\leq k_F,\sigma} \varepsilon_{\boldsymbol{k}}$ is the ground state energy of the Fermi gas (one assumes that the system is placed in a finite box of volume V with periodic boundary conditions).
(4 points)

(d) Show that the density $n = N/V$ is given, in the limit of large volume V, in terms of the Fermi momentum k_F, by $n = \dfrac{k_F^3}{3\pi^2}$ (in three dimensions).
(2 points)
Hint: Use $\frac{1}{V}\sum_{\boldsymbol{k}} \approx \frac{1}{(2\pi)^3}\int d^3k$ at large V.

(e) One has that $\varepsilon_{\boldsymbol{k}} = \dfrac{\hbar^2 k^2}{2m}$ for a non-relativistic system of massive fermions. Show that the energy density $e_0 = E_0/V$ is given as a function of n by $e_0 = \dfrac{(3\pi^2)^{5/3}\hbar^2}{10\pi^2 m} n^{5/3}$ (for large V).
(2 points)

(f) Finally, compute the chemical potential of the system $\mu = \dfrac{\partial e_0}{\partial n}$ in the ground state and show that $\mu = \dfrac{\hbar^2 k_F^2}{2m}$, *i.e.*, it is equal to the Fermi energy (at $T = 0$).
(2 points)

9 Dynamic structure factor for the Fermi gas

We consider a system of free fermions described by the Hamiltonian $\hat{H}_0 = \sum_{\boldsymbol{k},\sigma} \varepsilon_{\boldsymbol{k}} \hat{c}^\dagger_{\boldsymbol{k},\sigma}\hat{c}_{\boldsymbol{k},\sigma}$ with $\varepsilon_{\boldsymbol{k}} = \dfrac{\hbar^2 k^2}{2m}$. We wish to compute the value of $S(\boldsymbol{q},t) = \int_{-\infty}^{\infty} d\omega\, S(\boldsymbol{q},\omega)e^{-i\omega t}$, where the dynamic structure factor $S(\boldsymbol{q},\omega)$ is defined in (3.7) (see exercise sheet 1). Performing the integral over ω, one can show that $S(\boldsymbol{q},t)$ is given by

$$S(\boldsymbol{q},t) = \frac{1}{\hbar V}\big(\langle \hat{\rho}_{\boldsymbol{q}}(t)\hat{\rho}_{-\boldsymbol{q}}(0)\rangle - \langle \hat{\rho}_{\boldsymbol{q}}(t)\rangle\langle \hat{\rho}_{-\boldsymbol{q}}(0)\rangle\big), \tag{9.1}$$

where $\hat{\rho}_{\boldsymbol{q}}(t) = e^{i\hat{H}_0 t/\hbar}\hat{\rho}_{\boldsymbol{q}}e^{-i\hat{H}_0 t/\hbar}$ and where $\hat{\rho}_{\boldsymbol{q}}$ is given by equation (2.5) (see exercise sheet 1).

(a) Show that for a free fermion system $S(\boldsymbol{q},t) = \dfrac{1}{\hbar V}\sum_{\boldsymbol{k},\sigma} n_{\boldsymbol{k},\sigma}(1 - n_{\boldsymbol{k}+\boldsymbol{q},\sigma})e^{-i\zeta_{\boldsymbol{k},\boldsymbol{q}}t}$, where $\zeta_{\boldsymbol{k},\boldsymbol{q}} = \dfrac{1}{\hbar}\big(\varepsilon_{\boldsymbol{k}+\boldsymbol{q}} - \varepsilon_{\boldsymbol{k}}\big)$ and where $n_{\boldsymbol{k},\sigma} = \dfrac{1}{e^{\beta(\varepsilon_{\boldsymbol{k}}-\mu)}+1}$ is the Fermi-Dirac distribution.
(4 points)
Hint: Use the time evolution of the annihilation and creation operator in a system with a quadratic Hamiltonian and then apply the results of exercise 6c to compute the resulting average values.

(b) Show that $S(\boldsymbol{q}) = S(\boldsymbol{q}, t = 0)$ is strictly positive, as it should be.
(1 point)

(c) Performing the Fourier transform, one can trivially obtain

$$S(\boldsymbol{q}, \omega) = \frac{1}{\hbar V} \sum_{\boldsymbol{k},\sigma} n_{\boldsymbol{k},\sigma}(1 - n_{\boldsymbol{k}+\boldsymbol{q},\sigma})\delta(\omega - \zeta_{\boldsymbol{k},\boldsymbol{q}}). \tag{9.2}$$

Show that this expression can also be written as

$$S(\boldsymbol{q}, \omega) = \frac{1}{\hbar V} \sum_{\boldsymbol{k},\sigma} \frac{n_{\boldsymbol{k},\sigma}}{1 - e^{\beta\hbar\zeta_{\boldsymbol{k},\boldsymbol{q}}}} \Big[\delta(\omega - \zeta_{\boldsymbol{k},\boldsymbol{q}}) + e^{\beta\hbar\omega}\delta(\omega + \zeta_{\boldsymbol{k},\boldsymbol{q}})\Big], \tag{9.3}$$

which is very reminiscent of equation (4.1) for the single mode approximation, see exercise 4.
(4 points)
Hint: Use the identity $n_{\boldsymbol{k},\sigma}(1 - n_{\boldsymbol{k}+\boldsymbol{q},\sigma}) = \frac{n_{\boldsymbol{k},\sigma} - n_{\boldsymbol{k}+\boldsymbol{q},\sigma}}{1 - e^{\beta\hbar\zeta_{\boldsymbol{k},\boldsymbol{q}}}}$ (verify explicitly) in (9.2) and then perform an appropriate shift of the summation over $\boldsymbol{k}$ in one of the terms, taking into account that $\zeta_{-\boldsymbol{k}-\boldsymbol{q},\boldsymbol{q}} = -\zeta_{\boldsymbol{k},\boldsymbol{q}}$.

(d) Substitute (9.3) in (3.6) (see exercise sheet 1) and perform the integral over ω' to obtain

$$\chi(\boldsymbol{q}, \omega) = -\frac{2}{\hbar V} \sum_{\boldsymbol{k},\sigma} \frac{\zeta_{\boldsymbol{k},\boldsymbol{q}} n_{\boldsymbol{k},\sigma}}{(\omega + i\varepsilon)^2 - \zeta_{\boldsymbol{k},\boldsymbol{q}}^2}, \tag{9.4}$$

which is the *Lindhardt function.*
(2 points)

(e) Show that one has

$$\int_{-\infty}^{\infty} d\omega\, \omega\, S(\boldsymbol{q}, \omega) = \frac{1}{\hbar V} \sum_{\boldsymbol{k},\sigma} n_{\boldsymbol{k},\sigma}\zeta_{\boldsymbol{k},\boldsymbol{q}} = \frac{nq^2}{2m}, \tag{9.5}$$

and thus that the dynamical structure factor satisfies the *f-sum rule.*
(3 points)
Hint: Note that $n_{-\boldsymbol{k},\sigma} = n_{\boldsymbol{k},\sigma}$.