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Quantum Many-Body Systems

4. Exercise Sheet

13 The Baker-Campell-Hausdorff series

One defines the exponential of an operator $\hat{O}$ through the series $e^{\hat{O}} := \sum_{n=0}^{\infty} \frac{1}{n!}\hat{O}^n$, (for operators $\hat{O}$ that fulfill certain conditions)

(a) Show that

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \sum_{n=0}^{\infty} \frac{1}{n!}\hat{D}_n, \tag{13.1}$$

where $\hat{D}_0 = \hat{B}$ and $\hat{D}_{n+1} = [\hat{A}, \hat{D}_n]_-$ for $n \geq 0$.
(3 points)
Hint: Consider the operator $\hat{B}_\lambda = e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}}$ as a function of λ. Write down the equation for its derivative with respect to λ and substitute the generalization of (13.1) on both sides of such an equation. A comparison of the Taylor series arising on both sides of that equation is sufficient to obtain the desired result.

(b) Show that if $\hat{A}$ commutes with $[\hat{A}, \hat{B}]_-$, one has $e^{\hat{A}}e^{\hat{B}}e^{-\hat{A}} = e^{\hat{B}+[\hat{A},\hat{B}]_-}$
(2 points)

(c) Show that if the commutator $[\hat{A}, \hat{B}]_-$ commutes with both $\hat{A}$ and $\hat{B}$, one has $e^{\hat{A}}\hat{B} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]_-}$
(3 points)

Hint: Show that the derivative of $\hat{U}_\lambda = e^{-\lambda\hat{B}} e^{-\lambda\hat{A}} e^{\lambda(\hat{A}+\hat{B})+\frac{1}{2}\lambda^2[\hat{A},\hat{B}]_-}$ is identically zero if $[\hat{A},\hat{B}]_-$ commutes with both $\hat{A}$ and $\hat{B}$, the result then follows from considering the expression at $\lambda = 1$. You will need the result proven in **(a)**.

14 Coherent States of the harmonic oscillator

We consider the one-dimensional quantum harmonic oscillator, described by the Hamiltonian $\hat{H}_0 = \hbar\omega_0 \hat{a}^\dagger \hat{a}$ and by the commutator $[\hat{a}, \hat{a}^\dagger] = 1$. This system is equivalent to a single boson mode. We ignore the zero-energy contribution $\hbar\omega_0/2$ to the Hamiltonian, as it is a mere additive constant. It can be shown that such an Hamiltonian possesses normalized energy eigenstates $\hat{H}_0|n\rangle = E_n|n\rangle$ with $E_n = \hbar\omega_0 n$. It can also be shown that $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. In particular, the ground state of such a system, also known as the vacuum, satisfies $\hat{a}|0\rangle = 0$.

(a) Show that the partition function of such a system, defined as $Z_\beta = e^{-\beta\hbar\omega_0/2}\mathrm{Tr}\big(e^{-\beta\hat{H}_0}\big)$ (we re-introduced here the zero-point energy), is given by $Z_\beta = 1/(2\sinh(\beta\omega_0/2))$.
(2 points)
Hint: Express the trace in terms of the eigenstates $|n\rangle$ of $\hat{H}_0$.

(b) Check explicitly that the entropy of such a system satisfies the third law of thermodynamics, $S(T=0) = 0$, where $S = -\dfrac{dF}{dT}$ is the entropy of the system, with $F = -k_B T \ln Z_\beta$ being the free energy.
(2 points)

(c) One defines a coherent state $|z\rangle = \hat{T}(z)|0\rangle$, where z is a complex number and $\hat{T}(z) = e^{z\hat{a}^\dagger - z^*\hat{a}}$ is the so-called displacement operator (see lecture).

(i) Show that $|z\rangle = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^\infty \dfrac{z^n}{\sqrt{n!}}|n\rangle$.
(2 points)
Hint: Show first that $|z\rangle = e^{-\frac{1}{2}|z|^2} e^{z\hat{a}^\dagger}|0\rangle$, using the Baker-Campbell-Hausdorff identity, proved in the previous exercise.

(ii) Show that $\hat{a}|z\rangle = z|z\rangle$, *i.e.* $|z\rangle$ is an eigenstate of the (non-hermitian) operator $\hat{a}$.
(2 points)
Hint: There are at least two ways to solve this exercise. You may either consider the representation of $|z\rangle$ in terms of eigenstates of $\hat{H}_0$ and the action of $\hat{a}$ on these eigenstates, or you may want to use the previous exercise to compute the product $\hat{T}^\dagger(z)\hat{a}\hat{T}(z)$. Do note that $\hat{T}^\dagger(z)\hat{T}(z) = \mathbb{1}$, *i.e.*, $\hat{T}(z)$ is an unitary operator.

(iii) Show that the coherent states $|z\rangle$ and $|z'\rangle$ are normalized and that $\langle z'|z\rangle = e^{-\frac{1}{2}|z-z'|^2} e^{i\mathrm{Im}(zz'^\star)}$, where $z'^\star$ is the complex conjugate of z'. Do note that two coherent states characterized by different parameters z and z' are not

orthogonal.
(2 points)

(iv) Show that the time-evolution of $|z\rangle$ is given by $e^{-i\hat{H}_0 t/\hbar}|z\rangle = |ze^{-i\omega_0 t}\rangle$.
(2 points)
Hint: There are again two ways to solve this exercise. The first is to consider the action of $e^{-i\hat{H}_0 t/\hbar}$ in the representation of $|z\rangle$ in terms of eigenstates of $\hat{H}_0$, the second is to compute explicitly the product $e^{-i\hat{H}_0 t/\hbar}\hat{T}(z)e^{i\hat{H}_0 t/\hbar}$ using the known time-evolution of the creation and annihilation operators, see exercise 6.

(d) Show that the set of coherent states is complete by showing that $\int \frac{d\bar{z}dz}{2\pi i}|z\rangle\langle z| = \mathbb{1}$, where the measure $\frac{d\bar{z}dz}{2\pi i} = \frac{dxdy}{\pi}$, with x and y being the real and imaginary parts of z, *i.e.*, $z = x + i\,y$. The integrals over x and y extend from $-\infty$ to $+\infty$.
Since two coherent-states characterized by different parameters z and z' are not orthogonal, this basis is said to be *over-complete*.
(3 points)
Hint: Use the representation of $\hat{z}$ in terms of the eigenstates of $\hat{H}_0$ and the fact that such a basis is complete, *i.e.* $\sum_{n=0}^{\infty}|n\rangle\langle n| = \mathbb{1}$. Note that you will need to perform a transformation of the double integral over x and y to polar coordinates.

(e) Show that for any operator $\hat{O}$ (that fulfills certain conditions), one can write

$$\mathrm{Tr}\Big(\hat{O}\Big) = \int \frac{d\bar{z}dz}{2\pi i}\,\langle z|\hat{O}|z\rangle, \tag{14.1}$$

(2 points)
Hint: Express the trace of $\hat{O}$ in any complete orthogonal basis, *e.g.* the basis of the eigenstates of the Hamiltonian, and then use the (over-)completeness of the coherent state basis proved above, as well as the completeness of the said orthonormal basis, *i.e.* for the basis of eigenstates of the Hamiltonian such completeness is expressed as $\sum_{n=0}^{\infty}|n\rangle\langle n| = \mathbb{1}$.

(f) Show that $e^{-\beta\hat{H}_0}|z\rangle = e^{-\frac{1}{2}|z|^2(1-e^{-2\beta\omega_0})}|ze^{-\beta\hbar\omega_0}\rangle$
(3 points)
Hint: This exercise is solved in a manner that is completely analogous to (c)-iv, however, you have to be careful with the fact that $e^{-\beta\hat{H}_0}|z\rangle$ is not (unlike $e^{-i\hat{H}_0 t/\hbar}|z\rangle$) a normalized state, as $e^{-\beta\hat{H}_0}$ is not a unitary operator.

(g) Use the previous result and that of (c)-iii to show that $\langle z|e^{-\beta\hat{H}_0}|z\rangle = e^{-|z|^2(1-e^{-\beta\hbar\omega_0})}$. Show from such a result that

$$Z_\beta = e^{-\beta\hbar\omega_0/2}\int \frac{d\bar{z}dz}{2\pi i}\,\langle z|e^{-\beta\hat{H}_0}|z\rangle = \frac{1}{2\sinh(\beta\hbar\omega_0/2)}. \tag{14.2}$$

One thus recovers the result obtained in (a).
(4 points)
Hint: You will need to perform a coordinate transformation to polar coordinates in the resulting integral.

15 Path Integral representation of the quantum harmonic oscillator

The Green function of the 1d quantum harmonic oscillator is given by $G^R(x,T;x_0,0) = -\frac{i}{\hbar}e^{-i\omega_0 T/2}\langle x|e^{-i\hat{H}_0 T/\hbar}|x_0\rangle$ where $\hat{H}_0 = \hbar\omega_0 \hat{a}^\dagger\hat{a}$ and $T > 0$. It can be shown (see lecture course) that such a propagator has the following representation in terms of a path integral

$$G^R(x,T;x_0,0) = -\frac{i}{\hbar}\int \mathcal{D}[x(t)]\, e^{iS[x(t)]/\hbar}, \tag{15.1}$$

where the path integral is a sum over all pievewise continuous trajectories that start at $t = 0$ in point x_0 and end at $t = T$ in point x (*i.e.*, $x(0) = x_0$ and $x(T) = x$). The action $S[x(t)] = \frac{1}{2}m\int_0^T dt\,[\dot{x}^2(t) - \omega_0^2 x^2(t)]$ is computed over each of such trajectories (the action is said to be a *functional* of the trajectory) and each trajectory contributes with a different phase to the overall *sum over histories.* The objective of this exercise is to explicitly compute the propagator of the quantum harmonic oscillator.
Do note that we are going to perform heuristic calculations. If you want to see a path integral treated rigorously, see *e.g.* the book of Glimm and Jaffe, *Quantum Physics: A Functional Integral Point of View.*

(a) Show that $Z_\beta = e^{-\beta\hbar\omega_0/2}\mathrm{Tr}\left(e^{-\beta\hat{H}_0}\right) = i\hbar\int_{-\infty}^{\infty}dx\,G^R(x,-i\hbar\beta;x,0)$.
(2 points)
Hint:The basis of the position eigenstates $|x\rangle$ is complete.

(b) Writing $x(t) = x_c(t) + \delta x(t)$, where $x_c(t)$ is a solution of the classical equation of motion $\ddot{x}_c(t) = -\omega_0^2 x_c(t)$, with $x_c(0) = x_0$ and $x_c(T) = x$, and $\delta x(t)$ is the quantum fluctuation around such a solution with $\delta x(0) = \delta x(T) = 0$, show that one can write $S[x(t)] = S[x_c(t)] + S[\delta x(t)]$.
(3 points)
Hint: Substitute $x(t) = x_c(t)+\delta x(t)$ in $S[x(t)] = \frac{1}{2}m\int_0^T dt\,[\dot{x}^2(t)-\omega_0^2 x^2(t)]$, integrate one of the terms by parts and use the classical equation of motion for $x_c(t)$ and the boundary conditions for $\delta(t)$ at $t = 0$ and $t = T$.

(c) Show that $S[x_c(t)] = \frac{1}{2}m\int_0^T dt\,[\dot{x}_c^2(t) - \omega_0^2 x_c^2(t)]$ can be written as
$S[x_c(t)] = \frac{1}{2}m[x_c(T)\dot{x}_c(T) - x_c(0)\dot{x}_c(0)]$.
(1 point)

Hint: Integrate one of the points in $S[x_c(t)]$ by parts and use the classical equations of motion.

(d) Show that the solution of the classical equation of motion that satisfies the boundary conditions $x_c(0) = x_0$ and $x_c(T) = x$, is given by

$$x_c(t) = x_0 \cos(\omega_0 t) + \frac{x - x_0 \cos(\omega_0 T)}{\sin(\omega_0 T)} \sin(\omega_0 t). \tag{15.2}$$

(2 points)

(e) Substitute (15.2) in the expression given in (c) to show that

$$S[x_c(t)] = \frac{m\omega_0}{2\sin(\omega_0 T)} \left[x^2 \cos(\omega_0 T) - 2xx_0 + x_0^2 \cos(\omega_0 T)\right] \tag{15.3}$$

(2 points)

(f) Show from the results obtained above and from equation (15.1) that

$$G^R(x, T; x_0, 0) = -\frac{i}{\hbar} A(T) e^{iS[x_c(t)]/\hbar}, \tag{15.4}$$

where $A(T) = \int \mathcal{D}[\delta x(t)] e^{iS[x(t)]/\hbar}$ is an amplitude that is independent of the boundary conditions, as $\delta x(t)$ does not depend on them (recall that $\delta x(0) = \delta x(T) = 0$).
(2 points)

(g) Use (15.3) in (15.4), analytically continued to $T = -i\hbar\beta$, and substitute this result into (a) to obtain an expression for Z_β in terms of $A(-i\hbar\beta)$. Show from the known result for Z_β, as given by (14.2), that $A(-i\hbar\beta) = \sqrt{\frac{m\omega_0}{2\pi\hbar \sinh(\beta\hbar\omega_0)}}$.
(3 points)
Hint: You will need to perform a simple Gaussian integral.

(h) Finally, analytically continue this result by replacing $\hbar\beta \to iT$, to obtain

$$\begin{aligned} G^R(x, T; x_0, 0) = &-\frac{i}{\hbar}\sqrt{\frac{m\omega_0}{2\pi\hbar \sinh(\beta\hbar\omega_0)}} \\ &\times \exp\left[\frac{im\omega_0}{2\hbar \sin(\omega_0 T)} \left[x^2 \cos(\omega_0 T) - 2xx_0 + x_0^2 \cos(\omega_0 T)\right]\right]. \end{aligned} \tag{15.5}$$

This is the desired result for the propagator of the one-dimensional quantum harmonic oscillator. Note that the method employed here is only exact for linear systems.
(2 points)